

On a mixed problem in Diophantine approximation

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Abstract. *Let d be a positive integer. Let p be a prime number. Let α be a real algebraic number of degree $d+1$. We establish that there exist a positive constant c and infinitely many algebraic numbers ξ of degree d such that $|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < cH(\xi)^{-d-1} (\log 3H(\xi))^{-1/d}$. Here, $H(\xi)$ and $\text{Norm}(\xi)$ denote the naïve height of ξ and its norm, respectively. This extends an earlier result of de Mathan and Teulié that deals with the case $d = 1$.*

1. Introduction

In analogy with the Littlewood conjecture, de Mathan and Teulié [7] proposed recently a ‘mixed Littlewood conjecture’. For any prime number p , the usual p -adic absolute value $|\cdot|_p$ is normalized in such a way that $|p|_p = p^{-1}$. We denote by $\|\cdot\|$ the distance to the nearest integer.

De Mathan–Teulié conjecture. *For every real number α and every prime number p , we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0. \quad (1.1)$$

Obviously, the above conjecture holds if α is rational or has unbounded partial quotients in its continued fraction expansion. Thus, it only remains to consider the case when α is an element of the set ***Bad***₁ of badly approximable real numbers, where

$$\mathbf{Bad}_1 = \{\alpha \in \mathbf{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

De Mathan and Teulié [7] proved that (1.1) holds for every quadratic real number α (recall that such a number is in ***Bad***₁) but, despite several recent results [4, 3], the general conjecture is still unsolved.

If we rewrite (1.1) under the form

$$\inf_{a, q \geq 1, \gcd(a, q) = 1} q^2 \cdot \left| \alpha - \frac{a}{q} \right| \cdot |q|_p = 0,$$

then we have $|q|_p = \min\{|\text{Norm}(q/a)|_p, 1\}$. Hence, upon replacing α by $1/\alpha$, the de Mathan–Teulié conjecture can be reformulated as follows: For every irrational real number α , for every prime number p and every positive real number ε , there exists a non-zero rational number ξ satisfying

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-2}.$$

Throughout this paper, the height $H(P)$ of an integer polynomial $P(X)$ is the maximal of the absolute values of its coefficients. The height $H(\xi)$ of an algebraic number ξ is the height of its minimal defining polynomial over the rational integers $a_0 + a_1X + \dots + a_dX^d$, and the norm of ξ , denoted by $\text{Norm}(\xi)$, is the rational number $(-1)^d a_0/a_d$.

This reformulation suggests us to ask the following question.

Problem 1. *Let d be a positive integer. Let α be a real number that is not algebraic of degree less than or equal to d . For every prime number p and every positive real number ε , does there exist a non-zero real algebraic number ξ of degree at most d satisfying*

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-d-1}?$$

The answer to Problem 1 is clearly positive, unless (perhaps) when α is an element of the set ***Bad*** _{d} of real numbers that are badly approximable by algebraic numbers of degree at most d , where

$$\begin{aligned} \mathbf{Bad}_d = \{ \alpha \in \mathbf{R} : & \text{There exists } c > 0 \text{ such that } |\alpha - \xi| > c H(\xi)^{-d-1}, \\ & \text{for all algebraic numbers } \xi \text{ of degree at most } d \}. \end{aligned}$$

For $d \geq 1$, the set ***Bad*** _{d} contains the set of algebraic numbers of degree $d+1$, but it remains an open problem to decide whether this inclusion is strict for $d \geq 2$; see the monograph [2] for more information. The purpose of the present note is to give a positive answer to Problem 1 for every positive integer d and every real algebraic number α of degree $d+1$. This extends the result from [7] which deals with the case $d = 1$.

2. Results

Throughout this paper, for a prime number p , a number field \mathbf{K} , and a non-Archimedean place v on \mathbf{K} lying above p , we normalize the absolute value $|\cdot|_v$ in such a way that $|\cdot|_v$ and $|\cdot|_p$ coincide on \mathbf{Q} .

Our main result includes a positive answer to Problem 1 when α is a real algebraic number of degree $d+1$.

Theorem 1. *Let d be a positive integer. Let α be a real algebraic number of degree $d+1$ and denote by r the unit rank of $\mathbf{Q}(\alpha)$. Let p be a prime number. There exist positive constants c_1, c_2, c_3 and infinitely many real algebraic numbers ξ of degree d such that*

$$|\alpha - \xi| < c_1 H(\xi)^{-d-1}, \quad (2.1)$$

$$|\xi|_v < c_2 (\log 3H(\xi))^{-1/(rd)}, \quad (2.2)$$

for every absolute value $|\cdot|_v$ on $\mathbf{Q}(\xi)$ above the prime p , and

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < c_3 H(\xi)^{-d-1} (\log 3H(\xi))^{-1/r}. \quad (2.3)$$

Theorem 1 extends Théorème 2.1 of [7] that is only concerned with the case $d = 1$.

Under the assumptions of Theorem 1, Wirsing [10] established that there are infinitely many real algebraic numbers ξ satisfying (2.1).

The proof of Theorem 1 is very much inspired by a paper of Peck [8] on simultaneous rational approximation to real algebraic numbers. Roughly speaking, we use a method dual to Peck's to construct integer polynomials $P(X)$ that take small values at α , and we need an extra argument to ensure that our polynomials have a root ξ very close to α .

De Mathan [6] used the theory of linear forms in non-Archimedean logarithms to prove that Theorem 1 for $d = 1$ is best possible, in the sense that the absolute value of the exponent of $(\log 3H(\xi))$ in (2.2) cannot be too large. Next theorem extends this result to all values of d .

Theorem 2. *Let p be a prime number, d a positive integer and α a real algebraic number of degree $d+1$. Let λ be a positive real number. There exists a positive real number $\kappa = \kappa(\lambda)$ such that for every non-zero real algebraic number ξ of degree d satisfying*

$$|\alpha - \xi| \leq \lambda H(\xi)^{-d-1} \quad (2.4)$$

we have

$$|\xi|_v \geq (\log 3H(\xi))^{-\kappa}$$

for at least one absolute value $|\cdot|_v$ on $\mathbf{Q}(\xi)$ above the prime p .

As in [6], the proof of Theorem 2 rests on the theory of linear forms in non-Archimedean logarithms.

Let d be a positive integer. We recall that it follows from the p -adic version of the Schmidt Subspace Theorem that for every algebraic number α of degree $d+1$ and for every positive real number ε , there are only finitely many non-zero integer polynomials $P(X) = a_0 + a_1X + \dots + a_dX^d$ of degree at most d , with $a_0 \neq 0$, that satisfy

$$|P(\alpha)| \cdot |a_0|_p < H(P)^{-d-\varepsilon}.$$

Let ξ be a real algebraic number of degree at most d , and denote by $P(X) = a_0 + a_1X + \dots + a_dX^d$ its minimal defining polynomial over \mathbf{Z} . Then,

$$\min\{|\text{Norm}(\xi)|_p, 1\} \geq |a_0|_p$$

and there exists a constant $c(\alpha)$, depending only on α , such that

$$|P(\alpha)| \leq c(\alpha) H(\xi) \cdot |\xi - \alpha|.$$

Let ε be a positive real number. Applying the above statement deduced from the p -adic version of the Schmidt Subspace Theorem to these polynomials $P(X)$, we deduce that

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} \geq H(P)^{-d-1-\varepsilon}$$

holds if $H(P)$ is sufficiently large. This implies that if ξ satisfies (2.4) and if $H(\xi)$ is sufficiently large, then we have

$$|\text{Norm}(\xi)|_p \geq H(\xi)^{-\varepsilon},$$

accordingly

$$\max_{v|p} |\xi|_v \geq H(\xi)^{-\varepsilon/d}.$$

The result of Theorem 2 is more precise, however we cannot obtain a good lower bound for $|\text{Norm}(\xi)|_p$.

We conclude this section by pointing out that Einsiedler and Kleinbock [4] showed that a slight modification of the de Mathan–Teulié conjecture easily follows from a theorem of Furstenberg [5, 1].

Theorem EK. *Let p_1 and p_2 be distinct prime numbers. Then*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0$$

holds for every real number α .

In view of Theorem EK, we formulate the following question, presumably easier to solve than Problem 1.

Problem 2. *Let d be a positive integer. Let α be a real number that is not algebraic of degree less than or equal to d . For every distinct prime numbers p_1, p_2 and every positive real number ε , does there exist a non-zero real algebraic number ξ of degree at most d satisfying*

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_{p_1}, 1\} \cdot \min\{|\text{Norm}(\xi)|_{p_2}, 1\} < \varepsilon H(\xi)^{-d-1}?$$

Theorem EK gives a positive answer to Problem 2 when $d = 1$.

The sequel of the paper is organized as follows. We gather several auxiliary results in Section 3, and Theorems 1 and 2 are established in Sections 4 and 5, respectively.

In the next sections, we fix a real algebraic number field \mathbf{K} of degree $d + 1$. The notation $A \ll B$ means, unless specific indications, that the implicit constant depends on \mathbf{K} . Furthermore, we write $A \asymp B$ if we have simultaneously $A \ll B$ and $B \ll A$.

3. Auxiliary lemmas

Let \mathbf{K} be a real algebraic number field of degree $d + 1$. Let \mathcal{O} denote its ring of integers, and let $\alpha_0 = 1, \alpha_1, \dots, \alpha_d$ be a basis of \mathbf{K} . Let D be a positive integer satisfying

$$D(\mathbf{Z} + \alpha_1 \mathbf{Z} + \dots + \alpha_d \mathbf{Z}) \subset \mathcal{O} \subset \frac{1}{D}(\mathbf{Z} + \alpha_1 \mathbf{Z} + \dots + \alpha_d \mathbf{Z})$$

and the corresponding inequalities for the dual basis β_0, \dots, β_d defined by

$$\text{Tr}(\alpha_i \beta_j) = \delta_{i,j},$$

where Tr is the trace and $\delta_{i,j}$ is the Kronecker symbol.

We denote by $\sigma_0 = \text{Id}, \dots, \sigma_d$, the complex embeddings of \mathbf{K} , numbered in such a way that $\sigma_0, \dots, \sigma_{r_1-1}$ are real, $\sigma_{r_1}, \dots, \sigma_d$ are imaginary and $\sigma_{r_1+r_2+j} = \overline{\sigma_{r_1+j}}$ for $0 \leq j < r_2$. Set also $r = r_1 + r_2 - 1$, and let $\varepsilon_1, \dots, \varepsilon_r$ be multiplicatively independent units in \mathbf{K} .

Lemma 1. *Let η be a unit in \mathcal{O} such that $-1 < \eta < 1$ and define the real number N by $|\eta| = N^{-1}$. The conditions*

$$|\sigma_j(\eta)| \asymp N^{1/d}, \quad 0 < j \leq d, \quad (3.1)$$

and

$$|\sigma_i(\eta)| \asymp |\sigma_j(\eta)|, \quad 0 < i < j \leq d, \quad (3.2)$$

are equivalent. Let $\gamma \neq 0$ be in \mathbf{K} and let Δ be a positive integer such that $\Delta\gamma \in \mathcal{O}$. If η satisfies (3.1) or (3.2), write

$$\gamma\eta = a_0 + \dots + a_d \alpha_d,$$

with a_0, \dots, a_d in \mathbf{Q} . We have $D\Delta a_k \in \mathbf{Z}$ for $k = 0, \dots, d$ and

$$\max_{k=0, \dots, d} |a_k| \asymp N^{1/d},$$

where the implicit constants depend on γ .

Proof. Since η is a unit, we have

$$\prod_{0 \leq j \leq d} \sigma_j(\eta) = \pm 1.$$

and (3.1) and (3.2) are clearly equivalent. The formula

$$a_k = \text{Tr}(\gamma\eta\beta_k) = \gamma\eta\beta_k + \sum_{j=1}^d \sigma_j(\eta)\sigma_j(\gamma\beta_k)$$

implies that if η satisfies (3.1), then

$$|a_k| \ll N^{1/d}, \quad 0 \leq k \leq d.$$

Combined with

$$\sigma_1(\gamma)\sigma_1(\eta) = a_0 + \dots + a_d\sigma_1(\alpha_d),$$

this shows that

$$N^{1/d} \asymp |\sigma_1(\eta)| \ll \max_{k=0,\dots,d} |a_k|.$$

The proof of the lemma is complete. \square

Let α be a real algebraic number of degree $d+1$. We keep the above notation with the field $\mathbf{K} = \mathbf{Q}(\alpha)$ and the basis $1, \alpha, \dots, \alpha^d$ of \mathbf{K} over \mathbf{Q} , and we display an immediate consequence of Lemma 1.

Corollary 1. *Let η be a unit in \mathcal{O} such that $-1 < \eta < 1$ and set $N = |\eta|^{-1}$. Then*

$$D\Delta\gamma\eta = P(\alpha),$$

where $P(X)$ is an integral polynomial of degree at most d satisfying

$$H(P) \asymp N^{1/d}, \quad |P(\alpha)| \asymp N^{-1},$$

and thus

$$|P(\alpha)| \asymp H(P)^{-d}.$$

Denote by τ_j , $j = 0, \dots, d$ the embeddings of \mathbf{K} into \mathbf{C}_p . Recall that the absolute value $|\cdot|_p$ on \mathbf{Q} has a unique extension to \mathbf{C}_p , that we also denote by $|\cdot|_p$. In Lemmata 2 to 4 below we work in \mathbf{C}_p . Let $P(X)$ be an irreducible integer polynomial of degree $n \geq 1$. Let ξ be a complex root of $P(X)$ and ξ_1, \dots, ξ_n be the roots of $P(X)$ in \mathbf{C}_p . We point out that the sets

$$\{|\xi|_v : v \text{ is above } p \text{ on } \mathbf{Q}(\xi)\}$$

and

$$\{|\xi_i|_p : 1 \leq i \leq n\}$$

coincide, since all the absolute values $|\cdot|_v$ and $|\cdot|_p$ coincide on \mathbf{Q} .

Keeping the notation of Lemma 1, we have the following auxiliary result.

Lemma 2. *Assume that $\gamma = \alpha_d$. Then*

$$|a_k|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p, \quad 0 \leq k < d,$$

and

$$|a_d - 1|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p.$$

Proof. Since

$$\text{Tr}(\alpha_d\beta_k) = 0, \quad \text{for } k = 0, \dots, d-1,$$

we get

$$a_k = \text{Tr}(\gamma\eta\beta_k) = \text{Tr}(\alpha_d(\eta - 1)\beta_k) = \sum_{j=0}^d (\tau_j(\eta) - 1)\tau_j(\alpha_d\beta_k),$$

and deduce that

$$|a_k|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p, \quad 0 \leq k < d.$$

It follows from

$$\mathrm{Tr}(\alpha_d \beta_d) = 1$$

that

$$a_d = 1 + \mathrm{Tr}(\alpha_d \beta_d (\eta - 1)) = 1 + \sum_{j=0}^d (\tau_j(\eta) - 1) \tau_j(\alpha_d \beta_d),$$

and we derive that

$$|a_d - 1|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p.$$

This concludes the proof. \square

Lemma 3. *Let $0 < \delta < 1$. There exist arbitrarily large positive real numbers H and units η satisfying $\eta = H^{-d}$,*

$$\left| \frac{\sigma_j(\eta)}{\sigma_1(\eta)} - 1 \right| \leq \delta, \quad 2 \leq j \leq d, \quad (3.3)$$

and

$$|\tau_j(\eta) - 1|_p \ll (\log H)^{-1/r}, \quad 0 \leq j \leq d.$$

Proof. By taking suitable powers of the units $\varepsilon_1, \dots, \varepsilon_r$, we can assume that they are all positive, as well as their real conjugates, and that $|\tau_j(\varepsilon_i) - 1|_p < p^{-1/(p-1)}$ for $i = 1, \dots, r$ and $j = 0, \dots, d$. This is possible since $|\tau_j(\varepsilon_i)|_p = 1$ for $i = 1, \dots, r$ and $j = 0, \dots, d$. This allows us to consider the p -adic logarithms of each $\tau_j(\varepsilon_i)$. Our aim is to construct a suitable unit η of the form

$$\eta = \varepsilon_1^{\mu_1 p^s} \dots \varepsilon_r^{\mu_r p^s},$$

where $\mu_i \in \mathbf{Z}$. The conditions for (3.3) are then

$$p^s \left| \mu_1 \log \frac{|\sigma_j(\varepsilon_1)|}{|\sigma_1(\varepsilon_1)|} + \dots + \mu_r \log \frac{|\sigma_j(\varepsilon_r)|}{|\sigma_r(\varepsilon_r)|} \right| \leq C_1, \quad 2 \leq j \leq r,$$

where $C_1 = C_1(\delta) > 0$ is a constant, and

$$\left\| \frac{p^s}{2\pi} (\mu_1 \arg \sigma_j(\varepsilon_1) + \dots + \mu_r \arg \sigma_j(\varepsilon_r)) \right\| \leq C_2, \quad r_1 \leq j \leq r,$$

with $C_2 = C_2(\delta) > 0$. Set

$$Y_j = p^s \left(\mu_1 \log \frac{|\sigma_1(\varepsilon_1)|}{|\sigma_j(\varepsilon_1)|} + \dots + \mu_r \log \frac{|\sigma_1(\varepsilon_r)|}{|\sigma_j(\varepsilon_r)|} \right), \quad 2 \leq j \leq r,$$

and

$$Z_k = \frac{p^s}{2\pi} (\mu_1 \arg \sigma_k(\varepsilon_1) + \dots + \mu_r \arg \sigma_k(\varepsilon_r)) \in \mathbf{R}/\mathbf{Z}, \quad r_1 \leq k \leq r.$$

Taking $0 \leq \mu_i < M$, we have M^r points $(\mu_i)_{1 \leq i \leq r}$. The $(Y_j, Z_k)_{2 \leq j \leq r, r_1 \leq k \leq r}$ are in the product of intervals I_j , $2 \leq j \leq r$, of lengths $O(Mp^s)$ and of r_2 factors identical to \mathbf{R}/\mathbf{Z} . This set can be covered by $C_3(Mp^s)^{r-1}$ sets of diameter at most $\max\{C_1, C_2\}$, where C_3 is a constant depending on δ . By Dirichlet's *Schubfachprinzip*, choosing M such that

$$C_3(Mp^s)^{r-1} < M^r,$$

which can be done with

$$M \asymp p^{(r-1)s},$$

we get that there is $(\mu_1, \dots, \mu_r) \in \mathbf{Z}^r \setminus \{0\}$ such that

$$\max_{1 \leq i \leq r} |\mu_i| \ll M,$$

$$|Y_j| \leq C_1, \quad 2 \leq j \leq r,$$

and

$$\|Z_k\| \leq C_2, \quad r_1 \leq k \leq r.$$

Set then

$$\eta = (\varepsilon_1^{\mu_1} \dots \varepsilon_r^{\mu_r})^{p^s}$$

in such a way that $0 < \eta < 1$ (if needed, just consider $1/\eta$). This choice implies that

$$|\tau_i(\eta) - 1|_p = |\log_p \tau_i(\eta)|_p \leq p^{-s}, \quad 0 \leq i \leq d,$$

and

$$|\log \eta| \ll p^s M \ll p^{rs},$$

and the lemma is proved. \square

Lemma 4. *Let $P(X) \in \mathbf{C}_p[X]$ be a polynomial of degree d , and write*

$$P(X) = a_0 + \dots + a_d X^d.$$

Let ξ_i ($1 \leq i \leq d$) be the roots of $P(X)$ in \mathbf{C}_p . Let c be a real number satisfying $0 \leq c \leq 1$. If

$$|\xi_i|_p \leq c, \quad 1 \leq i \leq d,$$

we get

$$|a_k|_p \leq c|a_d|_p, \quad 0 \leq k < d. \quad (3.4)$$

Conversely, if (3.4) holds, then we have

$$|\xi_i|_p \leq c^{1/d}, \quad 1 \leq i \leq d.$$

Proof. Since

$$P(X) = a_d \prod_{1 \leq i \leq d} (X - \xi_i),$$

if $|\xi_i|_p \leq c \leq 1$ for $i = 1, \dots, d$, then we have

$$|a_k|_p \leq c|a_d|_p, \quad \text{for } k = 0, \dots, d-1.$$

Conversely, if

$$|a_k|_p \leq c|a_d|_p, \quad 0 \leq k < d,$$

and if $\xi \in \mathbf{C}_p$ is such that

$$a_d \xi^d + \dots + a_0 = 0,$$

then, there exists k with $0 \leq k < d$ and

$$|a_k \xi^k|_p \geq |a_d \xi^d|_p,$$

thus,

$$|\xi|_p^d \leq |\xi|_p^{d-k} \leq c.$$

This completes the proof of the lemma. \square

We conclude this section with two lemmas used in the proof of Theorem 2. The first of them was proved by Peck [8].

Lemma 5. *There exists a sequence $(\eta_m)_{m \geq 1}$ of positive units in \mathcal{O} such that*

$$\eta_m \asymp e^{-dm}$$

and

$$|\sigma_j(\eta_m)| \asymp e^m, \quad 1 \leq j \leq d.$$

Proof. Let us search the unit η_m under the form

$$\eta_m = \varepsilon_1^{\mu_1} \dots \varepsilon_r^{\mu_r},$$

with $\mu_i \in \mathbf{Z}$. We construct *real* numbers ν_1, \dots, ν_r such that

$$\nu_1 \log \varepsilon_1 + \dots + \nu_r \log \varepsilon_r = -dm \quad (3.5)$$

and

$$\nu_1 \log |\sigma_j(\varepsilon_1)| + \dots + \nu_r \log |\sigma_j(\varepsilon_r)| = m, \quad 1 \leq j \leq d. \quad (3.6)$$

Taking into account that, by complex conjugation, the equations (3.6) corresponding to an index j with $r_1 \leq j < r_1 + r_2$ and to the index $j + r_2$ are identical, and that the sum of (3.5) and equations (3.6) is zero, we simply have to deal with a Cramer system, since the matrix $(\sigma_j(\varepsilon_i))_{1 \leq j \leq r, 1 \leq i \leq r}$ is regular. We solve this system and then replace every ν_i by a rational integer μ_i such that $|\mu_i - \nu_i| \leq 1/2$. \square

Lemma 6. *Let λ' be a positive real number. Let $(\eta_m)_{m \geq 1}$ be a sequence of positive units as in Lemma 5. There exists a finite set $\Gamma = \Gamma(\lambda')$ of non-zero elements of \mathbf{K} such that for all integer polynomial $P(X)$ of degree at most d that satisfies*

$$|P(\alpha)| \leq \lambda' H(P)^{-d}, \quad (3.7)$$

there exist a positive integer m and γ in Γ for which

$$P(\alpha) = \gamma\eta_m.$$

Proof. Below, all the constants implicit in \ll depend on \mathbf{K} and on λ' . Let m be a positive integer such that

$$H(P) \asymp e^m,$$

and set

$$\gamma = P(\alpha)\eta_m^{-1}.$$

Since $D\alpha^k$ is an algebraic integer for $k = 0, \dots, d$, the algebraic number $D\gamma$ is an algebraic integer, and, by (3.7),

$$|\gamma| \ll 1.$$

Furthermore, for $j = 1, \dots, d$, we have

$$|\sigma_j(\gamma)| = |P(\sigma_j(\alpha))| \cdot |\sigma_j(\eta_m^{-1})| \ll H(P)e^{-m} \ll 1.$$

The algebraic integers $D\gamma \in \mathcal{O}$ and all their complex conjugates being bounded, they form a finite set. \square

4. Proof of Theorem 1

Let δ be in $(0, 1)$ to be selected later. Apply Lemma 3 with this δ to get a unit η and apply Lemma 1 with this unit and with $\gamma = \alpha^d$. Since $D^2\alpha^d\eta \in \mathbf{Z} + \dots + \alpha^d\mathbf{Z}$, we get

$$D^2\eta\alpha^d = a_0 + a_1\alpha + \dots + a_d\alpha^d = P(\alpha),$$

where, by Corollary 1, $P(X)$ is an integer polynomial of degree d and

$$|P(\alpha)| \asymp H(P)^{-d} \asymp H^{-d}.$$

By Lemma 2, each coefficient of $P(X)$ has its p -adic absolute value $\ll (\log 3H(P))^{-1/r}$, except the leading coefficient, whose p -adic absolute value equals $|D|_p^2$.

We then infer from Lemma 4 that all the roots of $P(X)$ in \mathbf{C}_p have their p -adic absolute value $\ll (\log 3H(P))^{-1/(dr)}$. This proves (2.2).

It now remains for us to guarantee that $P(X)$ has a root very close to α . To this end, we proceed to check that

$$|P'(\alpha)| \gg H(P).$$

Since

$$P'(\alpha) = a_1 + \dots + da_d\alpha^{d-1},$$

we get

$$P'(\alpha) = D^2 \left(\text{Tr}(\eta\alpha^d\beta_1) + 2\alpha\text{Tr}(\eta\alpha^d\beta_2) + \dots + d\alpha^{d-1}\text{Tr}(\eta\alpha^d\beta_d) \right),$$

hence,

$$P'(\alpha) = D^2 \sum_{i=0}^d \sum_{k=1}^d k \alpha^{k-1} \sigma_i(\eta \alpha^d \beta_k).$$

Let us write

$$P'(\alpha) = D^2 \sum_{i=0}^d A_i \sigma_i(\eta)$$

with

$$A_i = \sigma_i(\alpha^d) \sum_{k=1}^d k \alpha^{k-1} \sigma_i(\beta_k), \quad i = 0, \dots, d.$$

Observe first that

$$\sum_{i=1}^d A_i \neq 0.$$

Indeed, if this is not the case, then, working with the unit $\eta = 1$, that is, with $P(X) = D^2 X^d$ and $P'(\alpha) = dD^2 \alpha^{d-1}$, we get

$$d\alpha^{d-1} = A_0 = \alpha^d \sum_{k=1}^d k \alpha^{k-1} \beta_k,$$

hence,

$$d = \sum_{k=1}^d k \alpha^k \beta_k.$$

Taking the trace, and recalling that $\text{Tr}(\alpha^k \beta_k) = 1$, we get

$$d(d+1) = \sum_{k=1}^d k,$$

a contradiction.

Write

$$P'(\alpha) = D^2 \sum_{i=1}^d A_i \sigma_i(\eta) + O(H^{-d}) = D^2 \sigma_1(\eta) \sum_{i=1}^d A_i + B$$

with

$$|B| \leq D^2 \sum_{2 \leq i \leq d} |A_i| \cdot |\sigma_1(\eta)| \cdot \left| \frac{\sigma_i(\eta)}{\sigma_1(\eta)} - 1 \right| + O(H^{-d}).$$

Selecting now δ such that

$$\delta \sum_{2 \leq i \leq d} |A_i| \leq \frac{1}{3} \left| \sum_{i=1}^d A_i \right|,$$

we infer from Lemma 3 that

$$|P'(\alpha)| \geq \frac{1}{2}D^2 \left| \sigma_1(\eta) \sum_{i=1}^d A_i \right|,$$

when H is sufficiently large. This gives

$$|P'(\alpha)| \gg |\sigma_1(\eta)| \gg H.$$

Consequently, $P(X)$ has a root ξ such that

$$|\alpha - \xi| \ll H(P)^{-d-1} \ll H(\xi)^{-d-1}.$$

Classical arguments (see at the end of the proof of Theorem 2.11 in [2]) show that ξ must be real and of degree d if H is sufficiently large. This proves (2.1). Inequality (2.3) follows from (2.1) and (2.2) together with the fact that ξ is of degree d . This completes the proof of the theorem.

5. Proof of Theorem 2

The constants implicit in \ll and \gg below depend on \mathbf{K} , p and λ . There exists a positive real number λ' , depending on λ and on d , such that the minimal defining polynomial $P(X)$ of any real number ξ of sufficiently large height and for which (2.4) holds is of degree d and satisfies

$$|P(\alpha)| \leq \lambda' H(P)^{-d}.$$

Let $(\eta_m)_{m \geq 1}$ be as in Lemma 5. By Lemma 6, it is sufficient to prove Theorem 2 for the integer polynomials $P(X)$ as above such that

$$P(\alpha) = \gamma \eta_m = a_0 + a_1 \alpha + \dots + a_d \alpha^d.$$

Let ξ_i be the roots of $P(X)$ in \mathbf{C}_p and set

$$u := \max_{1 \leq i \leq d} |\xi_i|_p.$$

Assume that $u \leq 1$. It follows from Lemma 4 that

$$|a_k|_p \leq u |a_d|_p, \quad 0 \leq k < d.$$

Dividing $P(X)$ by $p^s = |a_d|_p^{-1}$ if necessary, we can assume that $|a_d|_p = 1$, and we obtain that

$$|a_k|_p \leq u, \quad 0 \leq k < d.$$

For $j = 1, \dots, d$, we then have

$$\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d}) = \sum_{k=0}^{d-1} a_k(\alpha^{k-d} - \tau_j(\alpha^{k-d})),$$

hence,

$$|\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d})|_p \ll u.$$

Since $|\eta_m|_p = 1$, we get that

$$\left| \frac{\tau_j(\eta_m)}{\eta_m} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1 \right|_p \ll u.$$

Upon writing

$$\eta_m = \varepsilon_1^{\mu_{1,m}} \dots \varepsilon_r^{\mu_{r,m}},$$

we have thus

$$u \gg \left| \left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1} \right)^{-\mu_{1,m}} \dots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r} \right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1 \right|_p$$

If

$$\frac{\tau_j(\eta_m)}{\eta_m} = \frac{\gamma\tau_j(\alpha^d)}{\tau_j(\gamma)\alpha^d}$$

holds for $j = 1, \dots, d$, the number

$$\gamma\eta_m\alpha^{-d}$$

is equal to all its conjugates, hence is rational, and we have

$$P(\alpha) = b\alpha^d$$

with $b \in \mathbf{Q}$, hence $P(X) = bX^d$, a contradiction. For every m , there thus exists an index j such that $1 \leq j \leq d$ and

$$\left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1} \right)^{-\mu_{1,m}} \dots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r} \right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} \neq 1.$$

Consequently, by the theory of linear forms in non-Archimedean logarithms (see e.g., Kunrui Yu [9]), there exists a positive constant κ such that

$$u \gg \left(\max_{1 \leq i \leq r} |\mu_{i,m}| \right)^{-\kappa}.$$

Since $\eta_m \asymp H(P)^{-d}$ and

$$|\log \eta_m| \asymp \max_{1 \leq i \leq r} |\mu_{i,m}|,$$

the matrix $(\log |\sigma_j(\varepsilon_i)|)_{1 \leq i \leq r, 1 \leq j \leq r}$ being regular, we conclude that

$$u \gg (\log 3H(\xi))^{-\kappa}.$$

This completes the proof of the theorem. □

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